

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 33, 149–162 (1971)

On Stochastic Boundedness and Stationary Measures for Markov Processes

L. E. N. DELBROUCK

University of Ottawa, Ottawa, Ontario, Canada

Submitted by K. Astrom

In this report we relate the property of stochastic boundedness to the existence of stationary measures for arbitrary Markov processes on the positive real line. We further develop a sufficiency criterion for the independence of such measures from initial conditions. The results are then applied to the question of approximating the fixed point vector of an irreducible infinite stochastic matrix by the solutions of finite ones.

INTRODUCTION

The transition operator for a certain discrete parameter Markov process on the interval $R = [0, \infty)$ has the following property. There is a set of states whence, on the average, transitions to the left are more heavily weighted. From the complement of this set, one jump transitions far to the right have uniformly small probabilities. It is shown herein that, under such conditions, non trivial invariant probability measures do exist and can be made independent of initial conditions through the addition of a weak-ergodicity criterion.

For the particular case of denumerable chains that are irreducible and aperiodic, one deduces the existence of a uniformly negligible "tail-end" set of states that may be forcibly prohibited. The evaluation of steady state probabilities then reduces to the question of finding the fixed point vector of an ordinary finite chain.

The analysis is based on the barest measure theoretic fundamentals, and perturbation-type arguments. Thereby, a greater measure of clarity and generality is achieved than is usually the case when complex variable theory is used.

This study was motivated by the analysis of service systems that exhibit little of the simple random walk properties that are customarily expected. It will be used as the theoretical groundwork for several practical problems such as the question of optimal allocation of waiting room space for mixed

loss-delay systems, and the troublesome problem of "warm-up" simulation runs in Monte-Carlo experiments for Markov chains with infinitely many states.

SUMMARY AND RESULTS

Let I , R , and B denote respectively the integer set $\{1, 2, \dots\}$, the interval $[0, \infty)$, and the family of Borel sets on R . For every $x \in R$, $G(x, \cdot)$ is a probability measure on $\{R, B\}$, such that for every $A \in B$, $G(\cdot, A)$ is B -measurable. In the usual way, we set $G_1(\cdot, \cdot) = G(\cdot, \cdot)$, and write

$$G_n(x, \cdot) = \int G_{n-1}(z, \cdot) G(x, dz) = \int G(z, \cdot) G_{n-1}(x, dz) \quad \text{for } n \geq 2.$$

In this paper, we investigate the existence and uniqueness of a solution for the integral equation $x(\cdot) = \int G(z, \cdot) x(dz)$ under the following:

Condition A. For some positive number m , there is an interval $C = [0, m]$ such that

$$(H1) \quad \sup_{z \in C} \sum_n \int_{\{x_i \in C; 1 \leq i \leq n\}} G(x_n, C) G(x_{n-1}, dx_n) \cdots G(z, dx) = a < \infty.$$

(e.g. $\sup_{z \in C} G_n(z, C) < 1$ for some $n \in I$);

(H2) Given any $\epsilon > 0$, there exists a positive number $N(\epsilon)$ such that

$$\sup_{z \in C} G(z, [N(\epsilon), \infty)) \leq \epsilon;$$

$$(H3) \quad \inf_{z \in R \sim C} \left\{ z - \int x G(z, dx) \right\} = w > 0.$$

Condition B. For every $\alpha \in R$

$$\sup_{\substack{A \in B \\ x, y \in [0, \alpha]}} |G(x, A) - G(y, A)| = \delta(\alpha) < 1$$

and $\delta(\infty) \leq 1$.

We show that under Condition A above there is in fact, a probability measure λ on $\{R, B\}$ which is a solution of the equation. Moreover, there is a calculable non-decreasing function f^* , with $f^*(\infty) = 1$, and $\lambda([0, x]) \geq f^*(x)$ for every $x \in R$.

If Condition B is added, one deduces the existence of a family of probability measures $\{\mu_\alpha\}_{\alpha \in R}$ such that μ_α is the unique stationary distribution of the Markov process generated by the kernel $G(\cdot, \cdot)$ but restricted to the

interval $[0, \alpha]$. The limit points of this family are solutions of the integral equation. Furthermore, for any pair α, β such that $m \leq \alpha \leq \beta$, the total variation of the signed measure $\mu_\alpha - \mu_\beta$ is dominated by the ratio $(1 - f^*(\alpha))/(1 - \delta(\beta))$. Hence if this ratio is asymptotically negligible as for example when $\delta(\infty) < 1$, then the solution of the integral equation is unique. Another bound is the minimum of the numbers $n(1 - f^*(\alpha)) + (\delta(\alpha))^{n-1}$ taken over $n \in I$.

We confine ourselves to the case $\delta(\infty) = 1$ which is frequently met in the applications. For $\delta(\infty) < 1$, the reader is referred to the theory of compact chains [1], [4]. Continuity conditions are not involved here and the discussion applies to discrete parameter processes of both continuous and discrete types as well. In particular we show that any aperiodic irreducible queueing process satisfying the hypothesis (H3) on the complement of a finite set must be positive recurrent. Its unique steady state probability vector may then be approximated coordinate-wise by that of a finite chain.

In conclusion, we establish the uniqueness of the invariant measure under Condition A and a weaker form of Condition B; namely, that for every positive real number α , there exists a positive integer n that may depend on α and such that for every $i \geq n$

$$\sup_{\substack{x, y \in [0, \alpha] \\ A \in B}} |G_i(x, A) - G_i(y, A)| < 1.$$

Finally, let it be understood once and for all that the notions of convergence and limit for measures are expressed in terms of the total variation norm, even though the distance function of Condition B is used throughout. There need be no ambiguity here since the former is not more than twice the latter.

2. THE INVARIANT MEASURES

Choose $\xi \in R$ arbitrarily and generate the sequence $\{X_j : j \in I\}$ satisfying $P(X_n \in A \mid X_0 = \xi) = G_n(\xi, A)$ for any $n \in I$. If $\xi \in \tilde{C}$, we write:

$$\begin{aligned} n(\xi) &= \min\{j : X_j \in C\}, \\ Z(\xi) &= \max\{X_1, X_2, \dots, X_{n^+(\xi)}\}, \end{aligned} \tag{1}$$

For $\xi \in C$, we set:

$$\begin{aligned} n^+(\xi) &= \min\{j : X_j \in \tilde{C}\}, \\ p(\xi) &= \min\{j : X_j \in C, j \geq n^+(\xi)\}, \\ Q(\xi) &= \max\{X_j : j \leq p(\xi)\}. \end{aligned} \tag{2}$$

Then we show

LEMMA 1. For any $\xi \in \tilde{C}$,

- (i) $P(n(\xi) = \infty) = 0$,
- (ii) $P(Z(\xi) \geq x) \leq \frac{2\xi}{x}, \quad x \neq 0$.

If $\xi \in C$,

- (iii) Given any $\epsilon > 0$, there exists a positive quantity $g(\epsilon)$ such that :

$$P(Q(\xi) \geq g(\epsilon)) \leq a\epsilon.$$

Proof. (i) Let

$$\begin{aligned} U_n &= X_n, & n \leq n^+(\xi), \\ &= X_{n(\xi)}, & \text{otherwise,} \\ W_n(\xi, A) &= P(U_n \in A \mid X_0 = \xi). \end{aligned}$$

Observe that:

$$\begin{aligned} W_n(\xi, \tilde{C}) &= P(X_i \in \tilde{C}; i = 1, 2, \dots, n \mid X_0 = \xi). \\ \lim_{n \rightarrow \infty} W_n(\xi, \tilde{C}) &= P(n(\xi) = \infty). \end{aligned}$$

By (H3), we have

$$\begin{aligned} E(U_n) &= \int_C x W_n(\xi, dx) \leq \int_C (x - w) W_{n-1}(\xi, dx) + \int_C x W_{n-1}(\xi, dx) \\ &= E(U_{n-1}) - w W_{n-1}(\xi, \tilde{C}), \\ w W_{n-1}(\xi, \tilde{C}) &\leq E(U_{n-1}) - E(U_n). \end{aligned}$$

Since clearly $\xi \geq E(U_{n-1}) \geq E(U_n)$, then

$$\lim_n W_{n-1}(\xi, \tilde{C}) = 0.$$

(ii) Write

$$\begin{aligned} Z_n(\xi) &= \max\{U_1, U_2, \dots, U_n\}, & Z(\xi) &= Z_{n(\xi)}(\xi), & Y_0 &= -\xi, \\ Y_n &= -U_n, & Y_n^* &= \min\{Y_1, Y_2, \dots, Y_n\} = -Z_n. \end{aligned}$$

Now, the sequence $\{Y_n\}_{n \in I}$ is a submartingale, and it must satisfy the following inequality, (see [1], formula (3.4"), p. 314)

$$-xP(Y_n^* \leq -x) \geq E(Y_1) - E(U_n) \geq -2\xi,$$

so that

$$P(Z_n(\xi) \geq x) \leq \frac{2\xi}{x}, \quad x \neq 0.$$

The last inequality, remains valid for the variable $Z(\xi)$ as well.

(iii) We write

$$\begin{aligned} \theta_0(x | \xi) &= 0, & \text{if } x < \xi, \\ &= 1, & \text{otherwise,} \\ \theta_1(x | \xi) &= G(\xi, [0, x]), \\ \theta_n(x | \xi) &= P(X_n \leq x; X_i \in C, i = 1, 2, \dots, n-1 | X_0 = \xi), \end{aligned}$$

so that for any $x \in R$,

$$P(Q(\xi) \geq x) = \sum_{n=1}^{\infty} \int_{x \in C} d\theta_{n-1}(x | \xi) \int_C P(Z(\sigma) \geq x) G(x, d\sigma). \quad (1.1)$$

Now choose $g(\epsilon) = 4\epsilon^{-1}N(2^{-1}\epsilon) > m$, and note by (H2) that

$$\int_C P(Z(\sigma) \geq g(\epsilon)) G(x, d\sigma) \leq \int_{m^+}^{N(2^{-1}\epsilon)} P(Z(\sigma) \geq g(\epsilon)) G(x, d\sigma) + 2^{-1}\epsilon. \quad (1.2)$$

On the other hand for $\sigma \in (m, N(2^{-1}\epsilon))$, we deduce from conclusion (ii):

$$P(Z(\sigma) \geq 4\epsilon^{-1}N(2^{-1}\epsilon)) \leq P(Z(\sigma) \geq 4\sigma\epsilon^{-1}) \leq \frac{2\sigma}{4\sigma\epsilon^{-1}} = 2^{-1}\epsilon. \quad (1.3)$$

By substitution in (1.1), conclude from (1.2), (1.3), and (H1) that

$$P(Q(\xi) \geq g(\epsilon)) \leq \epsilon \sum_{n=1}^{\infty} \theta_{n-1}(m | \xi) \leq \epsilon a. \quad (\text{q.e.d.})$$

Restated in simple terms, Lemma 1 asserts that any sample realization of a process generated by means of $G(\cdot, \cdot)$ must enter both the set C and its complement infinitely often; moreover it cannot (in the probability sense) assume arbitrarily large values on \tilde{C} . Specifically,

THEOREM 2. *For any $\xi \in R$, and given any $\epsilon > 0$, there exists a positive quantity $g''(\epsilon)$, and an integer $K(\cdot, \cdot)$ such that*

$$\sup_{j \geq K(\xi, \epsilon)} P(X_j \geq g''(\epsilon)) \leq \epsilon.$$

Hence there is a probability measure v_ϵ on $\{R, B\}$ such that

$$v_\epsilon(\cdot) = \int G(z, \cdot) v_\epsilon(dz).$$

Proof. Using $g(\epsilon)$ as defined in the previous lemma, let $g'(\epsilon) = g(a^{-1}\epsilon)$. For $\xi \in C$, choose $K(\xi, \epsilon) = 1$, $g''(\epsilon) = g'(\epsilon)$, and apply Lemma 1(iii) to verify the first assertion. If $\xi \in \bar{C}$, let $K(\xi, \epsilon)$ be the least integer j such that $P(n(\xi) \geq j) \leq 2^{-1}\epsilon$, and take $g''(\epsilon) = g'(2^{-1}\epsilon)$. Then for any integer $\alpha \geq K(\xi, \epsilon)$

$$\begin{aligned} P(X_\alpha \geq g''(\epsilon)) &= P(X_\alpha \geq g''(\epsilon) \mid n(\xi) < j) P(n(\xi) < j) \\ &\quad + P(X_\alpha \geq g''(\epsilon) \mid n(\xi) \geq j) P(n(\xi) \geq j) \\ &\leq 2^{-1}\epsilon P(n(\xi) < j) + P(n(\xi) \geq j) \leq \epsilon. \end{aligned}$$

In any case if $\alpha \geq K(\xi, \epsilon)$,

$$P(X_\alpha \geq g''(2^{-1}\epsilon)) = G_\alpha(\xi, [g'(2^{-1}\epsilon), \infty)) \leq \epsilon.$$

Hence the process $\{X_j : j \in I\}$ is stochastically bounded [2] whatever the starting point ξ . It follows by the Norms Ergodic Lemma [4] that there exists a subsequence of integers n_1, n_2, n_3, \dots such that the means

$$k^{-1} \sum_{i=1}^k G_{n_i}(\xi, \cdot)$$

tend to a probability measure v_ϵ satisfying

$$v_\epsilon(\cdot) = \int G(z, \cdot) v_\epsilon(dz)$$

$$v([0, g''(\epsilon)]) \geq 1 - \epsilon. \quad (\text{q.e.d.})$$

We shall now proceed to study the existence of limiting probability distributions independently of starting conditions.

3. THE CONSEQUENCES OF CONDITION B

Let it now be supposed that in addition to A , Condition B is also satisfied the restriction $\lim_x \delta(x) = 1$ being understood. Note that $\delta(\cdot)$ is a cumulative distribution on R , and so is the function f^* satisfying

$$f^*(g''(\epsilon)) = 1 - \epsilon. \quad (3)$$

For every $x \in R$ we further define,

$$\begin{aligned} R_x &= [0, x], \\ \chi_x(A) &= 1, \quad \text{if } x \in A, \\ &= 0, \quad \text{otherwise; } A \in B. \end{aligned} \quad (4)$$

In addition, we introduce the "truncated" kernel $G^{[x]}(\cdot, \cdot)$ satisfying

$$G^{[x]}(z, A) = G(z, A \cap R_x) + \chi_x(A) G(z, \tilde{R}_x), \quad z \in R, \quad A \in B. \quad (5)$$

Note that

$$\begin{aligned} G^{[x]}(z, A) &= G(z, A), \quad \text{if } A \subset [0, x], \\ G^{[x]}(z, R_x) &= 1. \end{aligned} \quad (6)$$

The iterates $G_n^{[\alpha]}(\cdot, \cdot)$ are defined in the obvious way.

For notational simplicity, we shall also define the following: for any two probability measures u , and v on $\{R, B\}$, write

$$\begin{aligned} L(u) &= \sup_{A \in B} \left| u(A) - \int G(z, A) u(dz) \right|, \\ d(u, v) &= \sup_{A \in B} |u(A) - v(A)|. \end{aligned} \quad (7)$$

Note in passing that

$$\int_R |u(dz) - v(dz)| \leq 2d(u, v), \quad (8)$$

and for any $A \in B$:

$$\begin{aligned} & \left| v(A) - \int G(z, A) v(dz) \right| \\ & \leq |v(A) - u(A)| + \int G(z, A) |u(dz) - v(dz)| + \left| u(A) - \int G(z, A) u(dz) \right| \end{aligned}$$

So that:

$$L(u) \leq 3d(u, v) + L(v). \quad (9)$$

LEMMA 3. For any $\alpha > m$, there is a unique probability measure μ_α on $\{R, B\}$ satisfying:

$$\begin{aligned} (i) \quad \mu_\alpha(\cdot) &= \int G^{[\alpha]}(z, \cdot) \mu_\alpha(dz), \\ \mu_\alpha([0, x]) &\geq f^*(x), \\ \mu_\alpha(R_\alpha) &= 1. \end{aligned}$$

Moreover, for any $\beta \geq \alpha$

$$\begin{aligned} \text{(ii)} \quad d(\mu_\alpha, \mu_\beta) &\leq (1 - \delta(\beta))^{-1} (1 - f^*(\alpha)), \\ d(\mu_\alpha, \mu_\beta) &\leq \min_{n \in I} \{n(1 - f^*(\alpha)) + (\delta(\alpha))^{n-1}\} \\ L(\mu_\beta) &\leq \{1 - f^*(\alpha)\}. \end{aligned}$$

Proof. (i) It is easily verified that

$$\sup_{\substack{A \in B \\ x, y \in R_\alpha}} |G^{[\alpha]}(x, A) - G^{[\alpha]}(y, A)| \leq \delta(\alpha). \quad (3.1)$$

Hence [5], there exists a probability measure $\mu_\alpha(\cdot)$ on $\{R, B\}$ with $\mu_\alpha(R_\alpha) = 1$ and such that

$$\begin{aligned} \mu_\alpha(\cdot) &= \int G^{[\alpha]}(z, \cdot) \mu_\alpha(dz), \\ \sup_{z \in R} d(\mu_\alpha, G_n^{[\alpha]}(z, \cdot)) &\leq [\delta(\alpha)]^{n-1}, \quad \text{for } n \in I. \end{aligned} \quad (3.2)$$

It may also be verified that the kernel $G^{[\alpha]}(\cdot, \cdot)$ satisfies Condition A with the same w , a , and $N(\cdot)$. Hence the conclusions of the previous section apply wholesale, and in particular the limiting measure μ_α is subjected to the probability bounds defined in Lemma 1, and Theorem 2.

(ii) Write

$$\begin{aligned} \gamma^{(1)}(\cdot) &= \int G^{[\alpha]}(z, \cdot) \mu_\beta(dz), \\ \gamma^{(n)}(\cdot) &= \int G^{[\alpha]}(z, \cdot) \gamma^{(n-1)}(dz) = \int G_n^{[\alpha]}(z, \cdot) \mu_\beta(dz). \end{aligned} \quad (3.3)$$

Observe that if we write

$$\epsilon(x) = \gamma^{(1)}([0, x]) - \mu_\beta([0, x]), \quad (3.4)$$

then

$$\begin{aligned} \epsilon(x) &= 0, & x < \alpha, \\ &= 1 - \mu_\beta([0, x]), & \text{otherwise.} \end{aligned}$$

Hence for any $A \in B$

$$\begin{aligned} \gamma^{(n)}(A) - \gamma^{(n-1)}(A) &= \int G_n^{[\alpha]}(z, A) d\epsilon(z) \\ &= \int_{z \geq \alpha} \{G_n^{[\alpha]}(\alpha, A) - G_n^{[\alpha]}(z, A)\} \mu_\beta(dz), \end{aligned} \quad (3.5)$$

and

$$|\gamma^{(n)}(A) - \gamma^{(n-1)}(A)| \leq \int_{z \geq \alpha} |G_n^{[\alpha]}(\alpha, A) - G_n^{[\alpha]}(z, A)| \mu_\beta(dz).$$

So that by conclusion (i) and (3.2)

$$\begin{aligned} d(\gamma^{(n)}, \gamma^{(n-1)}) &\leq [\delta(\beta)]^{n-1} (1 - f^*(\alpha)), \\ d(\gamma^{(n)}, \mu_\beta) &\leq (1 - f^*(\alpha)) \frac{1 - (\delta(\beta))^n}{1 - \delta(\beta)}, \end{aligned} \quad (3.6)$$

and

$$d(\mu_\alpha, \mu_\beta) \leq \lim_{n \rightarrow \infty} \{d(\gamma^{(n)}, \mu_\alpha) + d(\gamma^{(n)}, \mu_\beta)\} = \frac{1 - f^*(\alpha)}{1 - \delta(\beta)}.$$

The other bound results from (3.2) and from the fact that

$$d(\gamma^n, \gamma^{n-1}) \leq d(\gamma^{n-1}, \gamma^{n-2}).$$

Further and by (3.3) and (3.4)

$$\begin{aligned} \gamma^{(1)}(\{\alpha\}) &= \int G^{[\alpha]}(z, \{\alpha\}) \mu_\beta(dz) + \int G(z, \tilde{R}_\alpha) \mu_\beta(dz) \\ &= 1 - \gamma^{(1)}([0, \alpha)) \leq 1 - f^*(\alpha), \end{aligned} \quad (3.7)$$

whence, by definition and for any $A \in B$,

$$\begin{aligned} &\int G(z, A) \mu_\beta(dz) \\ &= \mu_\beta(A \cap [0, \alpha)) + \int G(z, A \cap \{\alpha\}) \mu_\beta(dz) + \int G(z, A \cap \tilde{R}_\alpha) \mu_\beta(dz) \\ &\leq \mu_\beta(A \cap [0, \alpha)) + (1 - f^*(\alpha)) \\ &= \mu_\beta(A) - \mu_\beta(A \cap [\alpha, \infty)) + (1 - f^*(\alpha)). \end{aligned} \quad (3.8)$$

The last assertion follows by conclusion (i). (q.e.d.)

Essentially then, as $\alpha \rightarrow \infty$, the effect of the operator $G^{[\alpha]}(\cdot, \cdot)$ on any one of the measures $\{\mu_\beta\}_{\beta \geq \alpha}$ becomes negligible. One must insure, however, that this effect remains small on successive iterations. More precisely, let

$$\begin{aligned} \psi(x, n) &= n(1 - f^*(\alpha)) + (\delta(\alpha))^{n-1}, \\ \theta(x) &= \sup_{z \geq x} (\inf_{n \in I} \psi(x, n)). \end{aligned} \quad (10)$$

Obviously $\psi < 2$ so that θ is well defined and non-increasing on R .

THEOREM 4. *There is a countable subset S of R , and a probability measure ν satisfying*

$$\lim_{\substack{\alpha \rightarrow \infty \\ \alpha \in S}} d(\nu, \mu_\alpha) = 0,$$

$$\nu(A) = \int G(x, A) \nu(dx), \quad A \in B,$$

and

$$\nu([0, x)) \geq f^*(x), \quad x \in R.$$

If moreover, $\lim_{x \rightarrow \infty} \theta(x) = 0$, then ν is the unique solution of the integral equation

$$x(\cdot) = \int G(x, \cdot) x(dx).$$

Conversely, if this solution is unique, then

$$\lim_{\substack{\alpha \rightarrow \infty \\ \beta \geq \alpha}} d(\mu_\alpha, \mu_\beta) = \lim_{\substack{\alpha \rightarrow \infty \\ \alpha \in R}} d(\mu_\alpha, \nu) = 0.$$

Proof. It is always possible to abstract from R a countable divergent sequence $S = a_1, a_2, a_3, \dots$ such that

$$\lim_{i \rightarrow \infty} (\sup_{j \geq i} d(u_{a_i}, u_{a_j})) = 0. \quad (4.1)$$

Because of Lemma 3(i), there then exists a probability measure ν on $\{R, B\}$ such that

$$\lim_{i \rightarrow \infty} d(\nu, u_{a_i}) = 0, \quad \nu([0, x)) \geq f^*(x). \quad (4.2)$$

By (9), however and for any $\beta \in S$:

$$L(\nu) \leq 3d(\mu_\beta, \nu) + L(\mu_\beta).$$

Then by (4.2) and Lemma 3(ii), we conclude that:

$$L(\nu) = 0.$$

Now the assumption $\theta(\infty) = 0$ means, by Lemma 3(ii) also, that

$$\lim_{\alpha \rightarrow \infty} (\sup_{\beta \geq \alpha} d(\mu_\alpha, \mu_\beta)) = 0$$

and hence:

$$\lim_{\substack{\alpha \rightarrow \infty \\ \alpha \in R}} d(\mu_\alpha, \nu) = 0.$$

The solution ν must then be unique because for any $\alpha > m$, μ_α is itself the unique solution of the equation

$$x(\cdot) = \int G^{[\alpha]}(z, \cdot) x(dz).$$

The concluding partial converse is also self-evident. (q.e.d.)

The question of convergence to a unique invariant measure will be more fully examined in the last section. But we shall first consider processes for which the uniqueness of the stationary distribution, if any, is not in question.

4. THE MARKOVIAN QUEUES

Let $P = [p_{ij}]$ be the transition matrix of a denumerable irreducible aperiodic chain. To be precise, p_{ij} denotes the probability of one step transition from j to i . As usual we write $P = P$, $P^n = P^{n-1}P = [p_{ij}^{(n)}]$, and we know from the characteristics of the chain that any non-trivial positive solution of the equation $x = Px$ must be unique. On the basis of the earlier discussion, one may assert

COROLLARY 5. *If for all but a finite set of states J , it is true that*

$$\inf_{j \notin J} \left\{ j - \sum i p_{ij} \right\} = w > 0,$$

then there is a unique non-trivial probability vector $\pi = \{\pi_0, \pi_1, \dots\}^T$ such that $\pi = P\pi$.

By way of proof we point out that this is simply a restatement of (H3) and that (H2) holds automatically since J is finite. Because of irreducibility there must then exist a $n \in I$ such that

$$\max_{j \in J} \sum_{i \in J} p_{ij}^{(n)} = h < 1,$$

so that (H1) holds as well.

The quantities $N(\epsilon)$ of assumption (H2) may be estimated from the asymptotic behavior of the step function $s(\cdot)$ defined by:

$$s(x) = \min_{j \in J} \sum_{i \leq x} p_{ij}.$$

Since clearly

$$\max_{j \in J} \sum_i \int_{x \in J} d\theta_i(x | j) \leq n(1 + h + h^2 \dots),$$

we may take $n/(1-h)$ as the constant a of assumption (H1), and proceed to construct a function f^* to estimate upper bounds for the "tail-end" coordinates of π .

Furthermore let ${}_kP = [{}_kp_{ij}]$ be the finite matrix satisfying

$${}_kp_{ij} = p_{ij}, \quad i = 0, 1, \dots, k-1, \quad j \leq k,$$

$${}_kp_{kj} = \sum_{i=k}^{\infty} p_{ij}, \quad j \leq k.$$

This matrix possesses a unique probability eigenvector

$$\pi(k) = (\pi_{k0}, \pi_{k1}, \dots, \pi_{kk})^T,$$

and for large k we know that the quantity $\max_{0 \leq i \leq k} |\pi_{ki} - \pi_i|$ is negligible. Moreover the vector $\pi(k)$ is obtainable by standard numerical methods of matrix powering.

This aspect of the discussion is of considerable practical importance especially in the case where severe dependence conditions on the transition mechanism rules out the application of transform techniques.

5. THE UNIQUENESS OF INVARIANT MEASURES

The probability bound $\psi(\cdot, \cdot)$ defined in Section 3 is based on the proposition that the total variation of finite signed measures is nondecreasing under stochastic linear transformation. For this reason, it will often be useless or at least pessimistic so that the assumption $\theta(\infty) = 0$ appears decidedly artificial. Hereafter, we place the question of convergence in a different frame of reference for in point of fact the Conditions A and B together exceed the requirements for uniqueness of the invariant measure.

Let us first remark that the Norms Ergodic Lemma mentioned in the proof of Theorem 2 actually states that the quantities $n^{-1} \sum_{i=1}^n G_i(\xi, \cdot)$ converge (for all $n \in I$) to an invariant measure. Since the proof of Theorem 2 was actually independent of the choice of ξ , it follows that any chain generated by means of the kernel $G(\cdot, \cdot)$ is weakly-regular in the sense defined by Shur [7]. It remains therefore to show how a weaker version of Condition B will, in conjunction with A, ensure that the chain is also strongly-regular ([7]; Definitions 2, 3, 4, Theorem 2). More precisely

THEOREM 6. *In addition to Condition A, suppose that for every $\alpha \in \tilde{C}$ there exists a $n \in I$ (which may depend on α) such that for every $i \geq n$*

$$\sup_{\substack{x, y \in R_\alpha \\ A \in B}} |G_i(x, A) - G_i(y, A)| < 1,$$

then there exists a unique probability measure λ on $\{R, B\}$ such that

$$\begin{aligned}\lambda(A) &= \int G(z, A) \lambda(dz), & A \in B, \\ &= \lim_{n \rightarrow \infty} G_n(\xi, A), & \text{any } \xi \in R.\end{aligned}$$

Proof. It follows from the hypothesis that the set R contains no proper Borel subset that forms a stochastically closed set of states so that R itself is a minimal-ergodic set relative to the kernel $G(\cdot, \cdot)$. In Shur's terminology, this together with the criterion of weak-regularity ensures that any chain generated by means of this kernel must be strongly regular. Conceivably, R might then contain cyclically moving subclasses, but again this must be ruled out by assumption. It follows that for every $x \in R$, $\lim_{n \rightarrow \infty} G_n(x, \cdot)$ is the only solution of the equation $x(\cdot) = \int G(z, \cdot) x(dz)$. (q.e.d.)

6. CONCLUDING REMARKS

In this article we have attempted to extend the theory of compact chains [4] to Markov processes on R that neither satisfy Doeblin type hypotheses nor lend themselves well to the traditional complex variable theoretic treatment. The hypotheses (H2) and (H3) appear to be mild requirements, at least insofar as the existence of non-trivial invariant measures is concerned. It is also doubtful that the condition (H1) might be significantly weakened. Be it pointed out incidentally that by retaining (H3), and substituting the condition $\sup_{z \in C} \int x G(z, dx) < \infty$, for (H1) and (H2), the existence of invariant measures is proven by a slight modification of an argument due to Foster [3]. Undoubtedly various combination of assumptions in conjunction with (H3) are possible. The Condition A, restricted to the case $\lim_{\alpha} \delta(\alpha) = 1$, rules out the partitioning of the set R into mutually exclusive ergodic sets. It is also commonly met in the applications involving infinite stochastic matrices with strictly positive elements or upper-right hand "corners" of zeros. If jointly applicable, the conditions A and B, provide for useful approximations, especially when the formulation and inversion of transforms prove to be computationally unattractive or even impossible.

REFERENCES

1. J. L. DOOB, "Stochastic Processes," Wiley, 1953.
2. W. FELLER, "An Introduction to Probability Theory and Its Applications," V. 2, Wiley, 1966.

3. F. G. FOSTER, On the Stochastic matrices associated with certain queueing processes, *Ann. Math. Stat.* **24** (1953), 335–360.
4. M. LOÈVE, “Probability Theory,” Van Nostrand, 1955.
5. S. V. NAGAEV, Some limit theorems for stationary Markov chains, *Theor. Probability Appl.* **2** (1957), 378–406.
6. N. U. PRABHU, “Queues and Inventories,” Wiley, 1965.
7. M. G. SHUR, Ergodic properties of invariant Markov chains on homogeneous spaces, *Theor. Probability Appl.* **3** (1958), 127–140.